Journal of Applied Mathematics and Mechanics 74 (2010) 188–197

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Similarity and dimensional analysis in the plane problem of the propagation of a vertical hydraulic fracture crack in an elastic medium☆

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ARTICLE INFO

Article history:
Received 26 June 2008

ABSTRACT

The problem of the growth of a vertical hydraulic fracture crack in an unbounded elastic medium under the pressure produced by a viscous incompressible fluid is studied qualitatively and by numerical methods. The fluid motion is described in the approximation of lubrication theory. Near the crack tip a fluid-free domain may exist. To find the crack length, Irwin's fracture criterion is used. The symmetry groups of the equations describing the hydraulic fracture process are studied for all physically meaningful cases of the degeneration of the problem with respect to the control parameters. The condition of symmetry of the system of equations under the group of scaling and time-shift transformations enables the self-similar variables and the form of the time dependence of the quantities involved in the problem to be found. It is established that at non-zero rock pressure the well-known solution of Spence and Sharp is an asymptotic form of the initial-value problem, whereas the solution of Zheltov and Khristianovich is a limiting self-similar solution of the problem. The problem of the formation of a hydraulic fracture crack taking into account initial data is solved using numerical methods, and the problem of arriving at asymptotic mode is investigated. It is shown that the solution has a self-similar asymptotic form for any initial conditions, and the convergence of the exact solutions to the asymptotic forms is non-uniform in space and time.

There are a large number of papers on various models of the evolution of a hydraulic fracture crack in an elastic deformable solid. The models by Perkins, Kern and Nordgren,[1,2] and Khristianovich, Geertsma and DeKlerk, [3,4] which are the most required in practices, have been used over many decades to calculate the opening and length of a hydraulic fracture crack. However, they are based on certain assumptions, which were made to simplify the solution of the problem and to obtain analytical results, that do not provide the possibility of taking into account all the specific features of the problem. Firstly, the elasticity problem for the surrounding medium is solved approximately, and the crack opening is found based on the solution for a uniformly loaded crack,[5] or assuming the local form of the relation between the fluid pressure and the crack opening (see Refs. 1,2,6,7). Secondly, the mutual consistency of the fluid flow problem and the problem of the deformation of the crack boundaries is not taken into account, and the fluid flow rate is assumed to be constant over the whole domain filled with fluid. Thirdly, the crack length is found from the material balance condition rather than from the fracture condition, i.e., the effects related to fracture energy losses are not taken into account.

By taking account of the consistency of the fluid pressure and the normal component of the medium stress on the crack boundary in the model, as well as a description of the fluid motion by the equations of lubrication theory,[8] a set of integro-differential equations is obtained that does not allow of an analytical solution.[9] If there is a finite flow rate in the crack, the pressure increases as the crack opening decreases and can become infinitely high when the opening tends to zero, i.e., at the crack tip. Such an infinite pressure cannot be realized physically, and hence, there must be a fluid-free domain near the crack tip.[3] In this case, the model must be completed with Irwin's fracture condition at the crack tip.[10]

Previously (see Refs. 9,10) the elements of similarity and dimensional methods were used to investigate various classes of self-similar solutions of the problem of hydraulic fracture crack growth. However, these methods were not implemented properly, in particular, the parameters that control the character of the solution were not indicated. A separation of variables was carried out and the control parameter K, which specifies the strength properties of an elastic medium, was only introduced for the part of the problem in which there

☆ *Prikl.Mat.Mekh.* Vol. 74, No. 2, pp. 262–275, 2010.

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0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2010.05.009

is a dependence on spatial variables, and this parameter was treated as the stress intensity factor.[9] As a result, in the total space-time problem the parameter K turns out to be time-dependent, and the solution obtained under the assumption $K \neq 0$ is therefore physically meaningless. The space variable of the problem was normalized using either the crack length, which is convenient when searching for self-similar solutions (although, in this case one has to solve a spatial problem), or a certain quantity that depends on time and has the dimension of length,[10] which has to be chosen by additional arguments of a physical nature. In the second case, two versions of normalization were used, namely, stiffness-like and viscosity-like ones. For each version its own set of control parameters was chosen, although, in fact, this set is unique, and all other sets are expressed in terms of it. The regime with vanishing rock pressure was treated as a degeneration of the problem with non-zero rock pressure when the dimensionless parameter with the meaning of time vanishes,[10] and this led to incorrect conclusions. In particular, it was stated that the crack begins to develop from a state described by the self-similar solution with vanishing rock pressure and a non-zero fluid-free domain. In this case it was assumed that in the course of time the fluid should fill a larger part of the crack volume and the crack should approach the state described by the Spence–Sharp self-similar solution.[9] However, the self-similar solution at zero rock pressure is itself asymptotic, and hence, further development of this regime is impossible.

In the present paper methods based on a symmetry analysis of the original equations of the problem with respect to similarity and time shift transformations are used. This enables us to study the features of the asymptotic form of the solution. It is shown that degeneration of the problem in control parameters proceeds with time for any choice of initial data. In the most important case of non-zero rock pressure it is shown that the asymptotic form of the solution is identical with the well-known result.[9]

The methods used previously,[9,10] which are based on solving the set of ordinary differential equations for the spatial part of the problem, do not enable the behavior of the problem with specified initial data to be investigated or the relation between the solutions to be found. As a consequence, in Refs. 9,10 many self-similar but physically meaningless solutions were analyzed. For example, it was pointed out (Ref. 10) that the asymptotic solution for zero rock pressure appears to be very close to the solution of the complete problem at non-zero rock pressure and short characteristic times, whereas in fact this solution is an intermediate asymptotic form and can only be realized under very specific conditions. For this case, the important result on the spatially non-uniform approach of physically meaningful solutions to the self-similar solution of Spence and Sharp (Ref. 9) was not formulated.

Hence, a group analysis of the original equations of the problem of the hydraulic fracture evolution first is carried out, self-similar solutions are constructed, and the problem of the transition to a self-similar regime of the solution in the initial-value problem is studied using numerical methods.

1. Statement of the problem

We will consider the model of a symmetric two-dimensional hydraulic fracture crack shown in Fig. 1. A crack develops in an infinite linearly elastic isotropic medium. The incompressible non-Newtonian fluid is pumped through the well, situated at the origin of coordinates. Fluid leakage to the surrounding medium is assumed to be small. The fluid pressure $P_f(x, t)$ given on the crack boundary and the compressing pressure $T_0 \leq 0$, specified at infinity, act on the elastic medium. The fluid fills only part of the crack volume $[-a(t), a(t)]$; near the crack tip there is a fluid-free domain.

The surrounding medium is linearly elastic, and therefore, a solution of the problem can be obtained in the form of the superposition of two problems, namely, the problem of the uniform stress-strain state of the surrounding medium without a crack under the action of the stresses, specified at infinity T_0, and the problem of a hydraulic fracture crack with zero stresses at infinity and at additional action on the crack boundary of the compressing rock pressure T_0. Henceforth, we will everywhere consider the statement of the second problem.

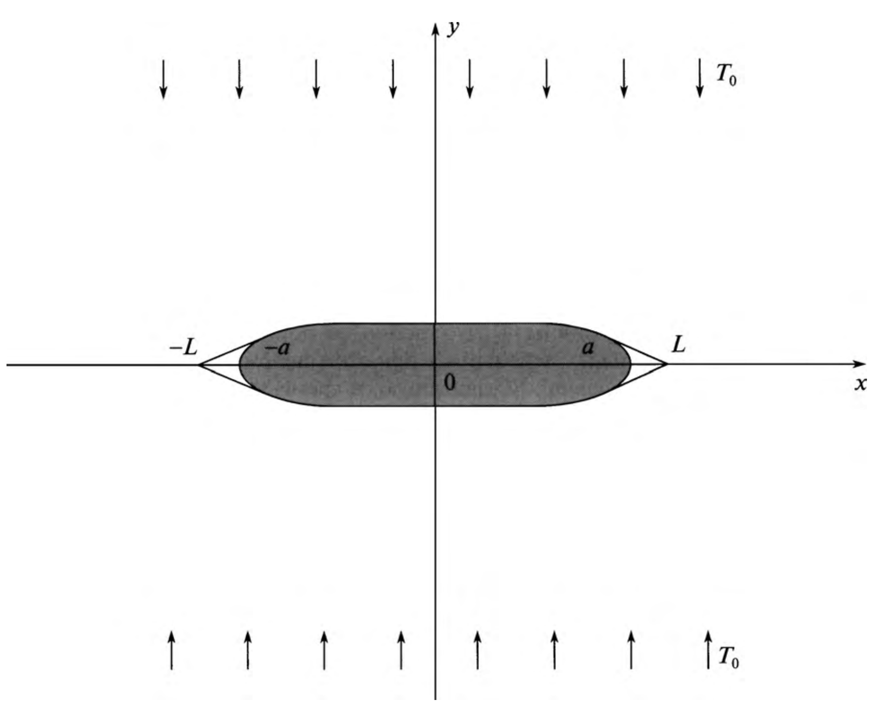

Fig. 1.

We nowturn to the mathematical statement of the problem. We will consider the well to be a point source situated at the origin of coordinates. The crack grows along the X axis and is symmetrical about the Y axis (see Fig. 1). The symmetry of the problem enables us to write the governing relations and the boundary conditions for half the crack $[0, L(t)]$.

The solution of the crack propagation problem consists of finding the half-length of the crack $L(t)$, the half-length of the domain filled with fluid $a(t)$, the crack opening $H(x, t)$, and the pressure at the crack boundary $P(x, t) = P_f(x, t) + T_0$ as functions of the x coordinate, the time t and the physical parameters of the problem, namely, the stiffness of the medium K_{Ic}, the elastic moduli E and ν, the fluid viscosity η, the magnitude of the rock pressure, and the flow rate of the fluid Q into the crack. Note that the elastic moduli in the problem only appear in the combination $E' = E/(1 + \nu^2)$, which corresponds to the assumption that the medium is plane-deformable.[11]

The pressure and the crack opening in the domain filled with fluid are related by the basic equation in the approximation of lubrication theory[8]

$$\frac{\partial H(x,t)}{\partial t} - \frac{1}{3\eta}\frac{\partial}{\partial x}\left(H^3(x,t)\frac{\partial P(x,t)}{\partial x}\right) = 0, \quad 0 \le x \le a \tag{1.1}$$

which is a consequence of continuity equation and the expression for the fluid flow rate in the crack described by Poiseuille's law

$$Q(x,t) = -\frac{H^3(x,t)}{3\eta}\frac{\partial P(x,t)}{\partial x} \tag{1.2}$$

The position of the fluid boundary at any time, $a(t)$, can be calculated from the equation of conservation of the injected fluid mass. In the local form the equation of mass conservation means that the fluid flow rate on the free boundary exactly determines the displacement of the boundary:

$$Q(a-0,t) = H(a-0,t)\frac{da}{dt} \tag{1.3}$$

Taking into account Eq. (1.2), expression (1.3) can be rewritten in the form

$$\frac{da}{dt} = -\frac{H^2(a-0,t)}{3\eta}\frac{\partial P(a-0,t)}{\partial x} \tag{1.4}$$

In the fluid-free domain the compressing rock pressure, which is the same at all points of this domain, acts on the crack:

$$P = T_0, \quad a \le x \le L \tag{1.5}$$

Because the pressure acting on the crack boundary is continuous along its whole length, the condition for the pressure to be constant in the fluid-free domain also yields the condition on the moving boundary.

The fluid flow rate at the crack inlet defines the boundary condition on the other boundary:

$$Q(0,t) = -\frac{H^3(x,t)}{3\eta}\frac{\partial P(x,t)}{\partial x}\bigg|_{x=0} = Q_0(t) \tag{1.6}$$

In the plane problem of the stress-strain state of the homogeneous isotropic elastic medium outside the crack, the pressure along the crack boundary $P(x, t)$ is connected with the crack opening $H(x, t)$ by the integral relation

$$P(x,t) = -\frac{E}{\pi(1-\nu^2)}\int_0^L \frac{\partial H(x',t)}{\partial x'}\frac{x'dx'}{x'^2 - x^2}, \quad 0 \le x \le L \tag{1.7}$$

The problem of the elastic state of the medium near the crack is treated as quasistatic one, and corresponds to the case when typical times of the development of dynamical processes in an elastic body are considerably shorter compared to the similar magnitudes for fluid flow in the crack. Thus, at each instant of time the crack fracture is in an equilibrium state and its length can be calculated from the condition that near the crack tip the stress intensity factor is equal to its critical value, that is, the stiffness of the medium:

$$K_I = K_{Ic} \tag{1.8}$$

Writing down the expression for the stress intensity factor in an elastically deformed solid, we obtain

$$K_{Ic} = 2\sqrt{\frac{L}{\pi}}\left[\int_0^a \frac{P(x,t)dx}{\sqrt{L^2 - x^2}} + T_0\int_a^L \frac{dx}{\sqrt{L^2 - x^2}}\right] \tag{1.9}$$

2. Introduction of dimensionless parameters

We will introduce the dimensionless variables

$$\xi = \frac{x}{a^*}, \quad \tau = \frac{t}{t^*}, \quad \bar{H} = \frac{H}{H^*}, \quad \bar{P} = \frac{P}{P^*}, \quad \bar{a} = \frac{a}{a^*}, \quad \bar{L} = \frac{L}{a^*}, \quad f = \frac{Q_0}{Q_0^*} \tag{2.1}$$

Here a^* is the characteristic length of the part of domain filled with fluid, t^* is the characteristic time of the hydraulic fracture process, H^* is the characteristic crack opening, $P^* = E'H^*/a^*$ is the characteristic pressure at the inlet, and $Q_0^* = \eta^{-1}(H^*)^3P^*/a^*$ is the characteristic magnitude of the fluid flow in the crack. Note also that in the problem under consideration the relation $H^*a^* = Q_0^*t^*$, which expresses the equality of the crack volume filled with fluid and the volume of fluid pumped, holds due to the smallness of the leakages.[12] In view of these relations we can specify in the arbitrary way only two characteristic parameters; their specific values can be chosen, for example, by constructing combinations of the physical parameters of the problem with appropriate dimensions.[9,10]

The dimensionless equations is written

$$\frac{\partial \bar{H}(\xi,\tau)}{\partial \tau} - \frac{1}{3}\frac{\partial}{\partial \xi}\left(\bar{H}^3(\xi,\tau)\frac{\partial \bar{P}(\xi,\tau)}{\partial \xi}\right) = 0 \quad \text{When} \quad 0 \le \xi \le \bar{a} \tag{2.2}$$

$$\frac{d\bar{a}}{d\tau} = -\frac{\bar{H}^2(\bar{a}-0,\tau)}{3}\frac{\partial P(\bar{a}-0,\tau)}{\partial \xi} \tag{2.3}$$

$$\bar{P}(\xi,\tau) = T \quad \text{When} \quad \bar{a} \le \xi \le \bar{L} \tag{2.4}$$

$$f(\tau) = -\frac{\bar{H}^3(\xi,\tau)}{3}\frac{\partial \bar{P}(\xi,\tau)}{\partial \xi} \quad \text{When} \quad \xi = 0 \tag{2.5}$$

$$\bar{P}(\xi,\tau) = -\frac{1}{\pi}\int_0^{\bar{L}}\frac{\partial \bar{H}(\xi',\tau)}{\partial \xi'}\frac{\xi' d\xi'}{\xi'^2 - \xi^2} \quad \text{When} \quad 0 \le \xi \le \bar{L} \tag{2.6}$$

$$\frac{\sqrt{\pi}}{2}K = \sqrt{\bar{L}}\left[\int_0^{\bar{a}}\frac{\bar{P}(\xi,\tau)d\xi}{\sqrt{\bar{L}^2 - \xi^2}} + T\int_{\bar{a}}^{\bar{L}}\frac{d\xi}{\sqrt{\bar{L}^2 - \xi^2}}\right] \tag{2.7}$$

The solution of the dimensionless system depends on the dimensionless spatial coordinate ξ, the dimensionless time τ, the dimensionless stiffness of the rock K, and the dimensionless rock pressure T. The parameters ξ, τ, K, and T are expressed in the form of combinations of dimensional quantities, in particular, it is convenient to express K and T in terms of Q_0^* and t^*

$$K = K_{Ic}\eta^{-1/4}E'^{-3/4}Q_0^{*\,-1/4}; \qquad T = T_0\eta^{-1/3}E'^{-2/3}t^{*1/3} \tag{2.8}$$

In the problem of the development of a hydraulic fracture, in addition to the quantities K and T, generally speaking, the control parameters which specify the initial data, for example, the pressure distribution along the crack and its length, must also be specified. However, when the hydraulic fracture develops, the initial length and width usually increase by a factor tens or even hundreds. It is obvious therefore that the dimensionless parameters describing the initial data will differ considerably from unity, and this may indicate that in this problem there is a self-similar asymptotic. The results of mathematical simulation show that after the initial crack length has doubled, the properties of the initial distribution turn out to be unimportant, that is, the solution in fact arrives at a self-similar regime.

Self-similar solutions often describe the asymptotic or intermediate-asymptotic behavior of the solution of the general problem, that is, after degeneration of the problem occurs in one or several parameters in the course of evolution, the solution takes a self-similar form.

From Eq. (2.8) it can be seen that for sufficiently long characteristic times the degeneration occurs in the parameter T, and, in the case of increasing pumping rate, in the parameter K. It turns out that when degeneration occurs in the parameters, the equations of the problem become symmetrical with respect to a certain transformation, the construction of which is given below. In particular, the use of symmetry properties enables us immediately to find the self-similar variables and to reduce the solution of the original problem to the solution of as set of ordinary differential equations.

3. Self-similar solutions

Sometimes dimension arguments turn out to be sufficient to establish the form of the self-similar variables. However, in the present problem their use does not enable appreciable simplifications to be obtained, though group considerations, a spacial case of which is the use of the similarity method, may be useful.

We will consider the similarity transformations of all the defined and defining parameters of the problem and choose in most general form those for which the equations of the problem remain unchanged

$$\xi' = A\xi, \quad \tau' = B\tau, \quad H' = DH, \quad P' = EP, \quad a' = F_2a, \quad L' = F_1L, \quad f' = Gf$$

The general form of the transformation may at once be restricted by taking $F_1 = F_2 = A$. Really, the form of the solution cannot be invariant under a transformation which changes the ratio of the length of the domain filled with fluid to the length of the whole crack, that is, $F_1/F_2 = 1$, and in addition, the parameter ξ takes values from the interval $[0, L]$, and hence, the transformation of the parameters ξ and L must leave their ratio unchanged, that is, $A/F_2 = 1$.

Note that, for a given arbitrary function f, i.e., the fluid flow rate at the crack inlet, the form of the solution cannot remain invariant under any transformation, since there is no information on how subsequent values of the function depend on preceding ones. In the case of similarity transformation we have $f = \tau^\alpha$, where α is an arbitrary real number.[13]

Now we can make the change

$$\xi' = A\xi, \quad \tau' = B\tau, \quad H' = DH, \quad P' = EP, \quad a' = Aa, \quad L' = AL, \quad f' = B^\alpha f$$

in Eqs. (2.2)–(2.7) and, in the case of nondegenerate values of parameters K and T, we obtain the conditions imposed on the similarity parameters from the requirement that the form of the solution under this transformation must be invariant:

$$\frac{D}{B}=\frac{D^3E}{A^2},\quad E=1,\quad \frac{D^3E}{A}=B^{\alpha},\quad E=\frac{D}{A},\quad \sqrt{A}E=1$$

This set of equations has no nontrivial solution. Thus, there is no similarity transformation that, in general form, leaves the form of the solution unchanged, and this means that in such a statement the problem does not allow of self-similar solutions. However, the number of relations for finding similarity parameters is reduced in the important practical case of degeneration in the parameters K and T.

We will first consider the case when there is degeneration in both these parameters. We then get a set of equations for finding the similarity parameters

$$\frac{D}{B}=\frac{D^3E}{A^2},\quad \frac{D^3E}{A}=B^{\alpha},\quad E=\frac{D}{A}$$

This set does not include the relations that were obtained before from Eqs. (2.4) and (2.7). These relations can be solved for B. Then

$$A=B^{\lambda},\quad D=B^{\mu},\quad E=B^{-1/3};\quad \lambda=\alpha/2+2/3,\quad \mu=\alpha/2+1/3$$

Thus, a group of similarity transformations exists that leaves the form of the solution invariant. The analysis by the similarity and dimension methods is a special case of the use of group methods. In particular, for the group of transformations just obtained the infinitesimal operator can be written as

$$\zeta\partial=\tau\partial_{\tau}+\lambda\xi\partial_{\xi}+\mu H\partial_H-\frac{1}{3}P\partial_P+\lambda a\partial_a+\lambda L\partial_L+\alpha f\partial_f \tag{3.1}$$

As it is well known, to construct the universal invariant J for the group of similarity transformations,[13] specified by its operator, it is necessary to solve the equation $\zeta\partial F=0$. In the case of operator (3.1) the solution of this equation reduces to finding the first integrals of a set of ordinary differential equations

$$\frac{d\tau}{\tau}=\frac{d\xi}{\lambda\xi}=\frac{dH}{\mu H}=\frac{dP}{(-1/3)P}=\frac{da}{\lambda a}=\frac{dL}{\lambda L}=\frac{df}{\alpha f} \tag{3.2}$$

As a result, the universal invariant can be written in the form

$$J=\left(\frac{\tau^{\lambda}}{\xi},\frac{\tau^{\mu}}{H},\frac{\tau^{-1/3}}{P},\frac{\tau^{\lambda}}{a},\frac{\tau^{\lambda}}{L},\frac{\tau^{\alpha}}{f}\right) \tag{3.3}$$

Using the universal invariant and writing the invariance condition for two different values of the time $\tau_1=\tau$ and $\tau_2=1$, we can rewrite the solution in self-similar variables

$$\begin{aligned}
&H(\xi,\tau)=\tau^{\mu}H(\xi/\tau^{\lambda},1)=\tau^{\mu}\tilde{H}(\xi/\tau^{\lambda})\\
&P(\xi,\tau)=\tau^{-1/3}P(\xi/\tau^{\lambda},1)=\tau^{-1/3}\tilde{P}(\xi/\tau^{\lambda})\\
&a(\tau)=\tau^{\lambda}a(1)=\tau^{\lambda}\tilde{a},\quad L(\tau)=\tau^{\lambda}L(1)=\tau^{\lambda}\tilde{L}
\end{aligned} \tag{3.4}$$

Consider the case when degeneration only occurs in the parameter T. Then the similarity coefficients must additionally satisfy the relation $\sqrt{A}E=1$, which follows from Eq. (2.7). Substituting the above expressions for the parameters A and E into the new relation, we obtain the condition $\alpha=0$, that is, in the case of degeneration only in the parameter T, a group of similarity transformations also exists, which leaves the form of the solution unchanged. Like the previous case, by using the universal invariant we can write the solution in terms of self-similar variables

$$\begin{aligned}
&H(\xi,\tau,K)=\tau^{1/3}\tilde{H}(\xi/\tau^{2/3},K),\quad P(\xi,\tau,K)=\tau^{-1/3}\tilde{P}(\xi/\tau^{2/3},K)\\
&a(\tau,K)=\tau^{2/3}\tilde{a}(K),\quad L(\tau,K)=\tau^{2/3}\tilde{L}(K)
\end{aligned} \tag{3.5}$$

Substituting the expressions for the desired quantities in terms of self-similar variables into Eqs. (2.2)–(2.7), we obtain a set of ordinary differential equations for finding $\tilde{H}$, $\tilde{P}$, $\tilde{a}$, and $\tilde{L}$.

Thus, using similarity and dimensional methods, we have establish two groups of transformations, undergo which the symmetry of the equations enables us to obtain two families of self-similar solutions. The equations are formally symmetric under the transformations from these groups only when the problem is degenerate in either the parameter T or both parameters K and T, thereby confirming the asymptotic character of the self-similar solutions of both families.

We will consider successively all possible regimes of crack growth for a power law for the rate of fluid pumping. If the fluid flow rate at the the crack inlet is kept at a constant level, conditions (2.8), $\alpha=0$ and $K\neq 0$, are satisfied. Then, for non-zero rock pressure the dimensionless parameter T, which specifies the relative value of the rock pressure, will increase, and, as a result, the fluid will occupy more and more of the cavity volume, and the solution will asymptotically tend to one of solutions (3.5). Formally, for the condition $T=0$, the family of self-similar solutions obtained earlier[10] for the first time and presented without a strict justification also arises. Unlike the self-similar solutions constructed for the condition $T=\infty$, the new solution includes the part of the fluid-free cavity. The ratio of the length of the domain filled with fluid to the total crack length remains constant and is governed by the parameter K. This solution will be asymptotic zero rock pressure. For some, fairly artificial, conditions, the self-similar solution obtained is an intermediate asymptotic of the solution for $T \neq 0$.

In fact, according to Eq. (2.8) the characteristic value of the pressure in the part of cavity filled with fluid is of the order of $P^* = (\eta/E')^{1/3}E'T^{*-1/3}$. Then, on the one hand, the dimensionless parameter that specifies the pressure distribution in the initial data obeys the condition $P_0/P^* \gg 1$, and consequently, the solution reduces to the asymptotic form when $t^* \gg ((\eta/E')E_0'^3)/P_0^3 = t_1$. On the other hand, the dimensionless value of the rock pressure is $T \ll 1$ when $t^* \ll ((\eta/E')E_0'^3)/T_0^3 = t_2$. This means that, if $T_1 \ll t_2$, the self-similar solution obtained under the condition $T = 0$ describes quite well the solution of the complete problem at times that are long enough for the influence of the initial peculiarities to disappear, and, along with that, are sufficiently small and we can neglect the fact that the value of the parameter T is non-zero. Further, when $t^* \gg t_2$ the parameter T tends infinity, and the solution takes the form (3.5). In such cases we say that the self-similar solution when $T = 0$ is an intermediate asymptotic.

If the flow rate at the crack inlet increases as a power law, i.e., $\alpha \neq 0$, the condition $K = 0$ is satisfied asymptotically. Then, at zero rock pressure the parameter T will tend to infinity, and with time the solution will take the form (3.4).

It has been pointed out[10] that when $T = \infty$ the self-similar solution obtained by Spence and Sharp[9] occurs, but in their problem statement the crack length is found from the equation of mass balance, rather then from the fracture criterion (2.7). The addition of condition (2.7) reduces the number of self-similar solutions of the system, apart from the case $K = 0$. The parameter K was treated as the stress intensity factor,[9] but this parameter was introduced after the separation of variables when studying the part of the problem, which depends only on the spatial variable. Hence, the parameter introduced by Spence and Sharp corresponds, in the original problem, a time-dependent stress intensity factor, which cannot be realized physically. In fact, the parameter K, as well as in the present paper, is a stress intensity factor normalized to the volume rate of fluid pumping into the crack. The condition of a power law pumping rate means that K decreases and tends to zero with time, and therefore, the self-similar solutions in Ref. 9 obtained for an increasing pumping rate and non-zero K are physically meaningless.

The previous considerations include all physically reasonable self-similar solutions. The other possibilities correspond to degenerate or unstable cases. The condition $K = \infty$ means a high resistance of the rock or a low fluid flow rate, which, in fact, means that the crack doesn't grow. The conditions $K = 0$ and $T = 0$ correspond to a low resistance of the rock or a high fluid flow rate for a low rock pressure; under these conditions the crack growth becomes unstable.

4. Limiting self-similar solutions

Consider the time-shift transformation $\tau' = \tau + \delta$, where δ is an arbitrary real number. Like in the case of the similarity transformation, the function f cannot be arbitrary. For the time-shift transformation we have $f = e^{\alpha t}$.[13] Transformations for the remaining quantities can be conveniently written in the form

$$\xi' = e^{A\delta}\xi,\quad H' = e^{D\delta}H,\quad P' = e^{E\delta}P,\quad a' = e^{A\delta}a,\quad L' = e^{A\delta}L$$

Here, like the case of the similarity transformation, the similarity coefficients for the quantities ξ, a and L are the same, since in the self-similar case they must vary in a matched manner.

Carrying out first of all the change in Eqs. (2.2), (2.3), (2.5) and (2.6), from the condition that the form of the solution should be invariant under the transformation, we obtain the conditions imposed on the similarity coefficients:

$$2D - 2A + E = 0,\quad D = A + E,\quad 3D + E - A = -\alpha$$

From this it follows that $E = 0$ and $A = D = -\alpha/2$. It should be noted that when $\alpha = 0$ the transformation degenerates; we therefore henceforth assume $\alpha \neq 0$.

The condition obtained from Eq. (2.4) imposes no additional limitations on the transformation parameters for any value of T, both finite or infinite.

Next, when $K \neq 0$ and $K \neq \infty$ the restriction $A/2 + E = 0$ obtained from Eq. (2.7) means that $A = 0$, and hence, a nontrivial self-similar solution does not exist. And, if $K = 0$, no additional restriction arises, and we finally have

$$A = D = -\alpha/2,\quad E = 0$$

Thus, when $K = 0$ a group of transformations exists that leaves the form of the solution unchanged. The infinitesimal operator of the group constructed can be written as

$$\zeta\partial = \partial_\tau + \frac{\alpha}{2}\xi\partial_\xi + \frac{\alpha}{2}H\partial_H + P\partial_P + \frac{\alpha}{2}a\partial_a + \frac{\alpha}{2}L\partial_L + \alpha f\partial_f \tag{4.1}$$

To construct the universal invariant of the group (see Ref. 14) we need to solve the equation $\zeta\partial F = 0$. Its solution reduces to finding the first integrals of the set of ordinary differential equations

$$d\tau = \frac{d\xi}{(\alpha/2)\xi} = \frac{dH}{(\alpha/2)H} = \frac{dP}{P} = \frac{da}{(\alpha/2)a} = \frac{dL}{(\alpha/2)L} = \frac{df}{\alpha f} \tag{4.2}$$

Hence, the universal invariant can be written as

$$J = \left(\frac{e^{\alpha\tau/2}}{\xi}, \frac{e^{\alpha\tau/2}}{H}, P, \frac{e^{\alpha\tau/2}}{a}, \frac{e^{\alpha\tau/2}}{L}, \frac{e^{\alpha\tau}}{f}\right) \tag{4.3}$$

Using the form of the universal invariant, we can rewrite the solution in terms of self-similar variables

$$\begin{aligned}
&H(\xi,\tau,T) = e^{\alpha\tau/2}H(\xi/e^{\alpha\tau/2},0,T) = e^{\alpha\tau/2}\tilde{H}(\xi/e^{\alpha\tau/2},T)\\
&P(\xi,\tau,T) = P(\xi/e^{\alpha\tau/2},0,T) = \tilde{P}(\xi/e^{\alpha\tau/2},T)\\
&a(\tau,T) = e^{\alpha\tau/2}a(0,T) = e^{\alpha\tau/2}\tilde{a}(T)\\
&L(\tau,T) = e^{\alpha\tau/2}L(0,T) = e^{\alpha\tau/2}\tilde{L}(T)
\end{aligned} \tag{4.4}$$

and from Eqs. (2.2)–(2.7) we obtain a set of ordinary differential equations for finding $\tilde{H}$, $\tilde{P}$, $\tilde{a}$ and $\tilde{L}$.

Thus, using the requirement that the set of equations must be invariant under a time shift transformation, the new family of self-similar solutions was found. The solutions of this family are formally realized both for $K = 0$ and $K = \infty$, but, as before, only the solution for $K = 0$ has a physical meaning. The condition $K = 0$ agrees with an exponential growth of the fluid flow rate at the well inlet and indicates the existence of the asymptotic solution belonging to this family. We also note a specific feature of this family, namely, the fact that the pressure is time-independent, i.e., in turn, this means that $T = T_0/P^*$=const. Another specific feature is the presence in the solution of a fluid-free domain of finite size for a finite value of $T \neq 0$.

In this case, the exponential solution is called a limiting solution for self-similar solutions of the usual form.[14]

In the case of a finite value of $T \neq 0$, the presence of a self-similarity of this form was pointed out previously,[3] and the exponential form of the growth in the flow rate was established, although only an approximate solution was considered. The case $T = 0$, like the case of a power law for the pumping rate, corresponds to unstable growth of the crack and has no physical meaning.

5. Numerical solution of the problem.

The numerical solution of the general problem with initial data enables us to verify the correctness of the conclusions on the asymptotic behaviour of the system. In fact, an analysis of the time behavior of the solution for various values of the problem parameters shows that this solution arrives at the self-similar asymptotic form.

In the numerical solution of the problem a number of difficulties arise. For example, when approximating Eq. (2.2), which describes the fluid flow in the crack, it is necessary to use an implicit difference scheme, since an approximation by an explicit scheme imposes considerable limits to the time step. Also note that when $\xi = \xi'$ in integral (2.6) there is a singularity, which has to be understood as a principal value in the Cauchy sense. It can be shown that within the framework of the linear approximation of the function H the value of integral tends to its value in the Cauchy sense while reducing the grid size, but one has to choose the grid in such a way that its points do not coincide with singular points of integral (2.6).

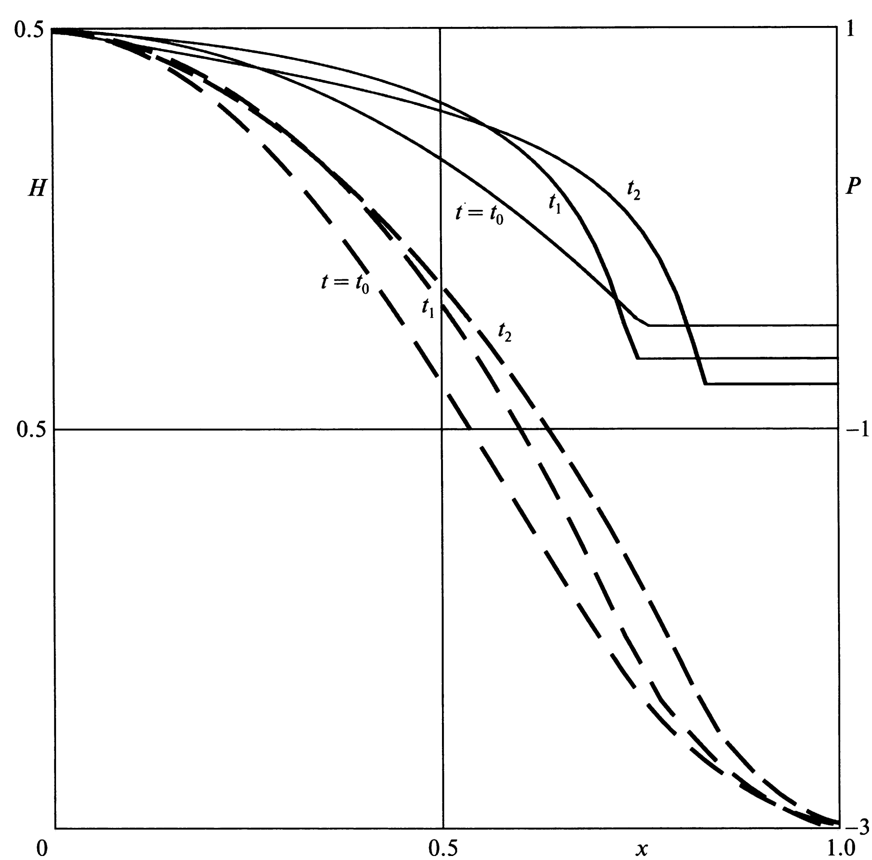

Fig. 2.

Since the statement of the problem assumes an agreement between the distribution of the crack opening H and the pressure P on the boundary, the need arises to determine the pressure from the opening (and vice versa) by using relation (2.6). Also, one has to be able to solve the problem of elasticity theory for an external rigid body for mixed boundary conditions when in the fluid filled domain the crack opening is specified and in the fluid-free domain the pressure is specified. In this case, to find the crack opening we have to solve an equation of the form

$$-\frac{1}{\pi}\int_a^L \frac{\partial H}{\partial x'}\frac{x'dx'}{x'^2 - x^2} = T + \frac{1}{\pi}\int_0^a \frac{\partial H}{\partial x'}\frac{x'dx'}{x'^2 - x^2}$$

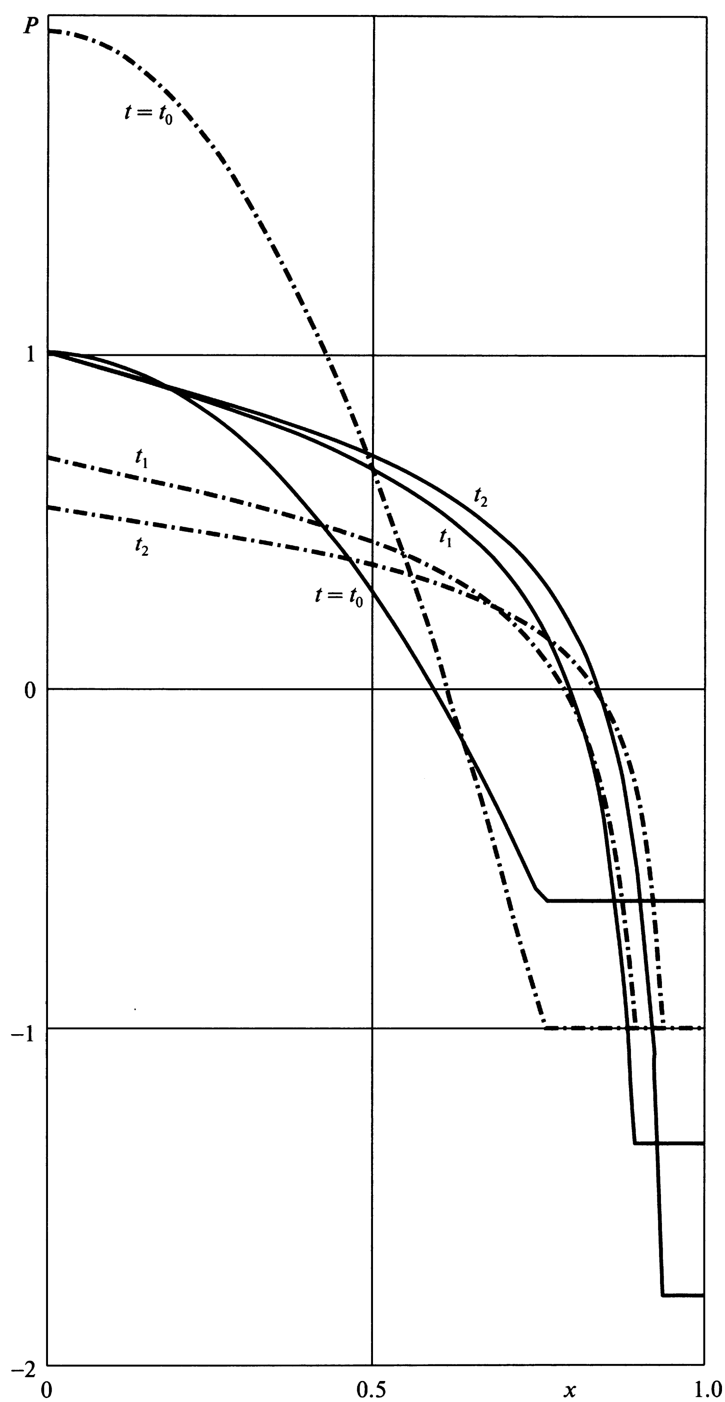

Fig. 3.

It is well known that the problem of finding the solution of the equation of this form is incorrect. However, integrating Eq. (2.6) by parts and taking into account the condition $H(L) = 0$, we obtain the equation

$$-\frac{1}{\pi}\int_a^L H\frac{x'^2+x^2}{(x'^2-x^2)^2}dx' = T + \frac{1}{\pi}\int_0^a H\frac{x'^2+x^2}{(x'^2-x^2)^2}dx'$$

that is suitable for the numerical solution of the problem.

Consider the example of the development of a crack when the pumping rate increases exponentially. Graphs of the pressure (the solid of curves) and for the crack opening (the dashed curves) at various instants of time, normalized along the vertical axis to the value of the pressure and crack opening at the crack inlet, and along the horizontal axis to the crack length, are shown in Fig. 2. Before beginning a calculation it is necessary to specify the size of the fluid-free domain and the pressure profile, and also to calculate the profile of the crack opening according to Eq. (2.6). In this example the initial pressure distribution in the fluid-filled domain is given by a parabola and corresponds to the curve for $t = t_0$ in Fig. 2. The crack opening calculated for such a pressure distribution also corresponds to the curve $t = t_0$. At the initial stage of modelling the redistribution of the fluid over the crack occurs, in the example chosen the fluid leaks into the crack more than at the initial instant of time, but one can specify such an initial fluid distribution that the redistribution will proceed in a more complex manner. We see that under the action of the injected fluid the pressure profile and the crack opening become steeper (the curves for t_1 and t_2 in Fig. 2). The size of the fluid-free domain, which corresponds to the horizontal segment in the graph of pressure distribution near the crack tip, decreases with respect to total crack length. The magnitude of the pressure in the fluid-free domain is given by a constant value $T < 0$, and the lowering of the horizontal segment in the pressure profile means a decrease in the pressure at the inlet crack. The solution very rapidly arrives at a self-similar regime (the curves for $t = t_2$); in this case the normalized profiles for the crack opening and pressure cease to change with time. The horizontal segment of the pressure profile ceases dropping, since, according to Eq. (4.4), the pressure at the crack intel becomes constant.

The most interesting case is when $T \neq 0$, that is, for non-zero rock pressure. In this case the self-similar regime with a finite length of the fluid-free domain is not established; the fluid tends to occupy the whole volume of the cavity. The variation of the pressure profiles when the crack is filling with fluid for this case is shown in Fig. 3. The pressure profiles, normalized to the value of pressure at the inlet to the crack, are represented by the solid curves, and those normalized to the absolute magnitude of the parameter T are represented by dot-dash curves. Comparison of the pressure profiles presented earlier[9] with those in Fig. 3 shows that the solution constructed in the present paper is closer to the self-similar solution obtained by Spence and Sharp, but non-uniformly in space. This result, which at first glance seems to be unexpected, is obvious. The solution constructed in the present paper when $T \neq 0$ in the variables normalized to the quantities $L(t)$ and $P(t)$ corresponds to the previously obtained solution[9] since as $t \to \infty$ the pressure at the crack tip $T = T_0/P(t)$ tends to infinity. Taking into account the fact that for finite and nonexponentially growing rates of fluid flow $P(t) \to 0$ as the crack opening grows, this solution is the most important in practics.

6. Conclusion

The specific feature of the present paper is, first, the use of classical methods of similarity and dimensional analysis, and second, the use of numerical methods in a general formulation of the problem of the development of a crack in a plane-deformed homogeneous medium. The generality of the statement due to by the fact that the crack size is calculated using Irwin's fracture criterion as well as the fact that the total joint problem is studied with a simultaneous analysis of the viscous fluid flow in the crack and the plane-deformed state of the surrounding. It it noteworthy that the use of numerical methods has enabled us to solve the problem with initial conditions and to investigate the arrival at an asymptotic regime; for such problems has been carried out here for the first time.

Using similarity and dimensional methods, self-similar asymptotic solutions have been obtained and the control parameters have been found. These are dimensionless values of the stress intensity factor and the rock pressure T. In this case, the parameter T tends to infinity as the characteristic time of the crack growth increases and the parameter K is constant for an unchanged value of the fluid flow rate at the crack intel, and tends to zero as the fluid flow rate increases.

It has been established by numerical experiments that the solution of the problem with initial conditions rapidly reaches the asymptotic form, and in the important practical case of non-vanishing rock pressure this is the asymptotic solution obtained by Spence and Sharp.[9] This is related to the obvious fact that $T(t) \to 0$ as $t \to \infty$ if the fluid flow rate does not increase exponentially. It is found that in the case of vanishing rock pressure, $T_0 = 0$, the self-similar solution is also asymptotic.

It has been established that the problem also admits of a limiting self-similar solution. In this case, to find the form of the self-similar variables, a group of time-shift transformations, which leaves the form of the solution unchanged, has been constructed. Essentially, the well-known solution by Khristianovich and Zheltov[3] is a limiting self-similar solution of this problem.

Acknowledgement

This research was financed by the Russian Foundation for Basic Research (09-01-00823).

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Translated by EVT